Many q-Particles from One: a New Approach to ∗-Hopf Algebras?

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LMU-TPW 94-15, hep-th/9411006, October 1994

Abstract

We propose a nonstandard approach to solving the apparent incompatibility between the coalgebra structure of some inhomogeneous quantum groups and their natural complex conjugation. In this work we sketch the general idea and develop the method in detail on a toy-model; the latter is a q-deformation of the Hopf algebra of 1-dim translations + dilatations. We show how to get all Hilbert space representations of the latter from tensor products of the fundamental ones; physically, this corresponds to constructing composite systems of many free distinct q-particles in terms of the basic one-particle ones. The spectrum of the total momentum turns out to be the same as that of a one-particle momentum, i.e. of the form $\{\mu q^n\}_{n\in\mathbf{Z}}$.

1 Introduction and preliminaries

The problem at the root of this work is the well-known apparent incompatibility between the Hopf structure of some quantum groups and their natural ∗-structure (i.e. complex conjugation). This incompatibility undermines the possibility of constructing composite quantum-mechanical systems with the corresponding q-group symmetry in terms of elementary ones. We have in mind especially q-deformations of inhomogenous Lie group such as Poincare's and the Euclidean one (which in the " undeformed " physics play an essential role as fundamental space(time) symmetries) in the form of braided semidirect products $IG_q := S_q \rtimes\!\!\!\lhd G_q$, [10, 9, 13, 3]; here G_q denotes an homogeneous quantum group and S_q the corresponding quantum space. When $q \in \mathbf{R}^+$ $Fun(IG_q)$ is a ∗-coalgebra but not a ∗-algebra, its dual $U_q(ig)$ is ∗-algebra but not a ∗-coalgebra (i.e. $(*\otimes*)\circ\phi \neq \phi\circ*$). When dealing with ∗-representations ρ of $U_q(ig)$ on Hilbert spaces, this implies in particular that if $\rho(O) \in \rho(U_q(ig))$ is an observable of a simple system, $(\rho\otimes\rho)\circ\phi(O)$ in general is *not* an observable of the composite system, since it is not hermitean.

The typical form of the coproduct of $U_q(ig)$ is

$$\phi(u) = u_i \otimes u'_i \qquad \phi(p^i) = p^i \otimes \mathbf{1} + \lambda u^i_j \otimes p^j \qquad u, u_i, u'_i, u^i_j \in U_q(g); \qquad (1.1)$$

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its restriction to the homogeneous sub-Hopf algebra $U_q(g)$ commutes with $*$, but $\phi, *$ don't commute on the translation generators p^i, due to the presence of the " dilaton " λ:

$$\lambda p^i = q^{-1} p^i \lambda, \qquad [\lambda, u] = 0, \qquad u \in U_q(g); \tag{1.2}$$

in fact, $(p^i)^* = p^j C_{ji}$ (C is a matrix of numbers) and $\lambda^* = \alpha\lambda^{-1}$ (with some $\alpha \in \mathbf{C}$). Nevertheless, $*$ maps the above coalgebra structure into a " conjugated " one, which is also compatible with the algebra structure of $U_q(ig)$.

The latter observation is at the hearth of our idea of using both " conjugated " coalgebras to form a genuine $*$-Hopf algebra isomorphic to $U_q(ig)$, and of how to use the latter in $*$-representation theory. Here we illustrate the idea in a most simple and pedagogical toy-model, representing a q-deformation of the abelian algebra of 1-dim translations (enlarged with dilatations) and of its Hilbert space representations. The spurious dilatation generator λ (which is absent in the undeformed Hopf algebra) is introduced to mimic the features of the Hopf-algebras $U_q(ig)$ summarized above. The algebra can be considered as the appropriate symmetry of some suitable q-deformed quantum mechanical systems consisting of one (or many) free particles on the quantum line $\mathbf{R}_q$. In a forthcoming paper [4] we are going to deal with the general problem sketched above and make contact with the work in Ref. [8].

We will assume that the reader is familiar with the axioms of bialgebras/ Hopf-algebras. Given a coassociative coproduct $\phi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, we will denote by $\phi_n : \mathcal{A} \to \bigotimes^n \mathcal{A}$ the map which can be obtained recursively in one of the following ways

$$\phi_{n+1} = \phi_{n,j}\phi_n, \qquad \phi_{n,j} := (\underbrace{id \otimes ... \otimes id}_{(n-j) \quad times} \otimes \phi \otimes \underbrace{id \otimes ... \otimes id}_{(j-1) \quad times}) \qquad 1 \leq j \leq n. \tag{1.3}$$

It will be useful to introduce the function

$$(a;q)_n := \prod_{i=0}^{n-1} (1 - aq^i), \qquad a \in \mathbf{C}, \quad n \in \mathbf{N}; \tag{1.4}$$

when $|q| < 1$ we will define $(a;q)_\infty := \lim_{n\to\infty}(a;q)_n$.

2 The toy-model

The building block of our toy-model consists of one free quantum particle on a quantum line $\mathbf{R}_q$, $q \in \mathbf{R}^+$, as considered in Ref. [5].

The starting algebra $\mathcal{A}$ is the unital $*$-algebra generated by elements $\{p^{\pm 1}, \lambda^{\pm 1}\}$ (actually we will assume that it is possible to extend it also to rational functions of the generators) satisfying the algebra relations

$$p\lambda = q\lambda p, \qquad \lambda^{\mp 1}\lambda^{\pm 1} = \mathbf{1} = p^{\mp 1}p^{\pm 1}, \tag{2.1}$$

and endowed with an antilinear involutive antihomomorphism $*$ (the " complex conjugation ") such that on the generators

$$(p^{\pm 1})^* = p^{\pm 1}, \qquad (\lambda^{\pm 1})^* = \lambda^{\mp 1}. \tag{2.2}$$

p plays the role of q-deformed translation generator in 1-dim configuration space; λ of q-deformed dilatation generator, or equivalently (since the model is one-dimensional), of q-deformed generator of boosts.

One can introduce a $*$-irrep (i.e. a $*$-irreducible representation) of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\mu$ in the following way. One postulates that $*$ also represents the operation of hermitean conjugation of operators and introduces an orthonormal basis of $\mathcal{H}_\mu$ $\{|n>\}_{n\in\mathbf{Z}}$ consisting of eigenfunctions of p:

$$p|n>=q^n\mu|n>, \tag{2.3}$$

where μ is a constant with dimension of a mass; p plays the role of q-deformed momentum observable. λ is a unitary " step " operator mapping eigenfunctions of p into each other:

$$\lambda^{\pm 1}|n>=|n\pm 1>. \tag{2.4}$$

For the sake of simplicity, here and in the sequel we will use the same symbols p,λ to denote both the abstract elements of $\mathcal{A}$ and their representations as Hilbert space operators on $\mathcal{H}_\mu$. Each vector $|n>$ is cyclic in $\mathcal{H}_\mu$, since $|m>=\lambda^{m-n}|n>$. Note that the state of rest $p=0$ does not belong to $\mathcal{H}_\mu$ (it belongs to some extension of $\mathcal{H}_\mu$, i.e. to the space of functionals on some dense domain $\mathcal{S}\subset\mathcal{H}_\mu$ of vectors " rapidly decreasing " at ∞), but can be approached as much as desired.

One can extend $\mathcal{A}$ to a q-deformed Heisenberg algebra $\mathcal{A}_H$ [5] by the introduction of a formally hermitean q-deformed position operator x defined [5] as a function of p,λ (see also Ref. [12]); a Hilbert space representation of $\mathcal{A}_H$ cannot be found on $\mathcal{H}_\mu$, but it can be found on its direct sum with the Hilbert space of opposite momentum, $\mathcal{H}_\mu\oplus\mathcal{H}_{-\mu}$. For the moment we will consider only irreps of $\mathcal{A}$, i.e. systems with momentum having a well defined sign.

Next, we choose the Hopf algebra structure $\{\phi,\epsilon,S\}$ for our toy model; this is needed to construct many-particle systems by using one particle ones as building blocks.

One possibility is the classical (in the sense of undeformed) one. The actions of $\phi_c,\epsilon_c,\sigma_c$ on the generators have the form

$$\phi_c(p)=p\otimes\mathbf{1}+\mathbf{1}\otimes p \qquad \phi_c(\lambda^{\pm 1})=\lambda^{\pm 1}\otimes\lambda^{\pm 1} \qquad \phi_c(\mathbf{1})=\mathbf{1}\otimes\mathbf{1} \tag{2.5}$$

$$\epsilon_c(p)=0 \qquad \epsilon_c(\lambda^{\pm 1})=1=\epsilon_c(\mathbf{1}) \tag{2.6}$$

$$\sigma_c(p)=-p \qquad \sigma_c(\lambda^{\pm 1})=\lambda^{\mp 1} \qquad \sigma_c(\mathbf{1})=\mathbf{1}, \tag{2.7}$$

and are extended to all of $\mathcal{A}$ as algebra homomorphisms (ϕ,ϵ) or antihomomorphism (σ) respectively. The physical meaning of any Hopf structure $\{\phi,\epsilon,\sigma\}$ for $\mathcal{A}$ should be the following. ϕ should allow to represent the algebra relations (2.1) on the Hilbert space $\mathcal{H}$ of states of a system composed of two subsystems; $\mathcal{H}$ should be built as the tensor product of the Hilbert spaces $\mathcal{H}_1,\mathcal{H}_2$ of the two subsystems. For instance, $\phi(p)$ should be an observable and should represent the momentum operator of the composite system, $\phi(\Lambda)$ the corresponding step operator. $\epsilon(p),\epsilon(\Lambda)$ should give the eigenvalues of the operators p,Λ on the vector with $p=0$. $\sigma(p)$ should represent the " opposite " momentum operator of p w.r.t. the " sum " operation introduced by ϕ, in the sense that $(id\otimes\sigma)\circ\phi(p)$ and $(\sigma\otimes id)\circ\phi(p))$ should represent the relative momenta of each subsystem in the rest frame of the other; the constraint $m\circ(id\otimes\sigma)\circ\phi(p)=0$ $(m(a\otimes b):=ab)$, which is a consequence of the axioms of an Hopf algebra and of definition $\epsilon(p)=0$, would guarantee that the relative momentum of one system w.r.t. its own rest frame is zero.

The Hopf structure (2.5-7) is compatible with the definition (2.2) of $*$ in the sense that $(*\otimes *)\circ\phi_c=\phi_c\circ *$ etc. It is immediate to verify that the spectrum of $\phi(p)$ on the tensor product of two Hilbert spaces $\mathcal{H}_{\mu_a}\otimes\mathcal{H}_{\mu_b}$ *doesn't* coincide with the spectrum of p either on $\mathcal{H}_{\mu_a}$,

or on $\mathcal{H}_{\mu_b}$. However, the definition of ϕ_c automatically fulfills the physical requirement that in any normalized eigenvector $|\psi>_{1\otimes 2}=|n>_1 \otimes|m>_2$ of $\phi_c(p)$, $\mathbf{1}\otimes p$

$$|<\psi|\phi(p)|\psi>_{1\otimes 2}| \; > \; \begin{cases} |<\psi|\mathbf{1}\otimes p|\psi>_{1\otimes 2}| \\ |<\psi|p\otimes \mathbf{1}|\psi>_{1\otimes 2}| \end{cases} \qquad if \;\; \frac{\mu_a}{\mu_b}>0 \tag{2.8}$$

namely the the total momentum is greater than the modulus of the momentum of either subsystem, if they have momenta with the same sign.

We are rather interested in a different Hopf structure, which more closely mimics the ones of actual inhomogeneous quantum groups $U_q(ig)$, and essentially presents the same kind of incompatibility with the *standard* $*$-structure. The corresponding ϕ,ϵ,S act on the generators according to the relations

$$\phi(p)=p\otimes\mathbf{1}+\lambda\otimes p \qquad \phi(\lambda^{\pm 1})=\lambda^{\pm 1}\otimes\lambda^{\pm 1} \qquad \phi(\mathbf{1})=\mathbf{1}\otimes\mathbf{1} \tag{2.9}$$

$$\epsilon(p)=0 \qquad \epsilon(\lambda^{\pm})=1=\epsilon(\mathbf{1}) \tag{2.10}$$

$$\sigma(p)=-\lambda^{-1}p \qquad \sigma(\lambda^{\pm 1})=\lambda^{\mp 1} \qquad \sigma(\mathbf{1})=\mathbf{1}, \tag{2.11}$$

and is extended to functions of p,λ in the same standard way as before. The above is not compatible with the standard $*$-structure, since $(*\otimes *)\circ\phi\neq\phi\circ *$. Nevertheless, it satisfies the following nonstandard properties

$$(*\otimes *)\circ\phi=\bar{\phi}\circ *, \qquad *\circ\epsilon=\bar{\epsilon}\circ * \qquad \sigma\circ *\circ\bar{\sigma}\circ *=id \tag{2.12}$$

Here $\{\bar{\phi},\bar{\epsilon},\bar{\sigma}\}$ is the Hopf structure which is obtained by the replacement $\lambda\to\lambda^{-1}$ in both sides of equations (2.5-7), i.e.:

$$\bar{\phi}=p\otimes\mathbf{1}+\lambda^{-1}\otimes p \qquad \bar{\phi}(\lambda^{\pm 1})=\lambda^{\pm 1}\otimes\lambda^{\pm 1} \qquad \bar{\phi}(\mathbf{1})=\mathbf{1}\otimes\mathbf{1} \tag{2.13}$$

$$\bar{\epsilon}(p)=0 \qquad \bar{\epsilon}(\lambda^{\pm 1})=1=\bar{\epsilon}(\mathbf{1}) \tag{2.14}$$

$$\bar{\sigma}(p)=-\lambda p \qquad \bar{\sigma}(\lambda^{\pm 1})=\lambda^{\mp 1} \qquad \bar{\sigma}(\mathbf{1})=\mathbf{1}, \tag{2.15}$$

(note that $\bar{\epsilon}=\epsilon$).

The two Hopf structures $\{\phi,\epsilon,\sigma\}$, $\{\bar{\phi},\bar{\epsilon},\bar{\sigma}\}$ are the bosonizations of the two basic braided Hopf structures associated to the quantum line $\mathbf{R}_q$ (making it a " braided line " [7]), with braiding respectively given by

$$\psi(p\underline{\otimes}p)=q^{-1}p\underline{\otimes}p \qquad \bar{\psi}(p\underline{\otimes}p)=qp\underline{\otimes}p. \tag{2.16}$$

An analogous pair of conjugated (braided) Hopf algebras arises also for quantum spaces which are comodule algebras of a bialgebra $A(R)$ [1], where R is a solution of the Yang-Baxter equation (see Ref. [4]).

3 The nonstandard $*$-realization of the Hopf algebra $\{\mathcal{A},\phi,\epsilon,\sigma\}$

It is our (philosophical) viewpoint that the barred and unbarred Hopf structures (or equivalently their braided versions) should be considered as two faces of the same medal, since the underlying homogenous q-group G_q symmetry is the same, and we hope to report soon on this point at a more abstract level elsewhere [4].

Here we are going to show in the case of the toy model how to construct from their *pair* an actual $*$-Hopf algebra, and how the latter should be used in representation theory on Hilbert spaces; as an Hopf algebra, it is isomorphic to either the unbarred or the barred version. Our arguments will be rather heuristic in order to motivate the construction.

To implement the abovementioned viewpoint we start from the trivial observation that $\gamma := \bar{\phi} \circ \phi^{-1}$ is an algebra isomorphism of $\phi(\mathcal{A})$ onto $\bar{\phi}(\mathcal{A})$ with the property

$$(* \otimes *) \circ \gamma = \gamma^{-1} \circ (* \otimes *), \tag{3.17}$$

due to equation $(2.12)_1$; moreover if we define $\Phi(a)$ as the *pair* $\Phi(a) := (\phi(a), \bar{\phi}(a))$ and define a multiplication $\Phi(a)\Phi(b) := \Phi(ab)$, the set $\Phi(\mathcal{A})$ gets the same algebra structure as $\mathcal{A}$, $\Phi(A) \approx \mathcal{A}$. Then setting

$$*_2 := \tau \circ ((* \otimes *), (* \otimes *)) \tag{3.18}$$

defines an antilinear involutive antihomomorphism on $\Phi(\mathcal{A})$, as a consequence of property (2.12); here $\tau[(A, B)] := (B, A)$. $*_2$ commutes with Φ, in the sense that $*_2 \circ \Phi = \Phi \circ *$; as a consequence, Φ maps real (w.r.t. $*_2$) elements of $\mathcal{A}$ into real elements of $\Phi(A)$.

This suggests that we look for an extension of the algebra isomorphism γ to all of $\mathcal{A} \otimes \mathcal{A}$ preserving relation (3.18), so that $*_2$ can be extended to an antilinear involutive antihomomorphism of the whole algebra $(\mathcal{A} \otimes \mathcal{A})_d := \{(\alpha, \gamma(\alpha)) \mid \alpha \in \mathcal{A} \otimes \mathcal{A}\}$ onto itself. Here the multiplication of$(\mathcal{A} \otimes \mathcal{A})_d$ is defined by $(\alpha, \gamma(\alpha)) \cdot (\beta, \gamma(\beta)) := (\alpha \cdot \beta, \gamma(\alpha \cdot \beta))$, so that $(\mathcal{A} \otimes \mathcal{A})_d \approx \mathcal{A} \otimes \mathcal{A})$. Given an element $A \in (\mathcal{A} \otimes \mathcal{A})_d$, we will introduce its decomposition in unbarred and barred components by $A =: (\rho_2(A), \bar{\rho}_2(A))$; consequently, $\gamma = \bar{\rho}_2 \circ \rho_2^{-1}$ and $(* \otimes *) \circ \rho_2 = \bar{\rho}_2 \circ (* \otimes *)$.

If this is achieved, then, we can reiterate the construction for higher tensor products, defining $\gamma_n := \bar{\phi}_n \circ (\phi_n)^{-1}$, $\Phi_n(a) := (\phi_n(a), \bar{\phi}_n(a))$ and antilinear involutive antihomomorphisms $*_n$ $(n \geq 1)$ through

$$*_n := (\rho_n(*_n), \bar{\rho}_n(*_n)) := \tau \circ \left(\bigotimes^n *, \bigotimes^n * \right), \tag{3.19}$$

($\phi_2 \equiv \phi$, $\bar{\phi}_2 \equiv \bar{\phi}$, $\gamma_2 \equiv \gamma$, $*_1 \equiv *$, $\gamma_1 \equiv id$) first on $\Phi_n(A)$, then by an extension of γ_n possibly on the whole $(\bigotimes^n \mathcal{A})_d := \{(a, \gamma_n(a)) \mid a \in \bigotimes^n \mathcal{A}\} \approx \bigotimes^n \mathcal{A}$ (the multiplication in $(\bigotimes^n \mathcal{A})_d$ is defined by $(a, \gamma_n(a)) \cdot (b, \gamma_n(b)) := (a \cdot b, \gamma(a \cdot b)))$, satisfying the fundamental property that it commutes with $\Phi_n := (\phi_n, \bar{\phi}_n)$, in the sense that $*_n \circ \Phi_n = \Phi_{n-1} \circ *_{n-1}$. If now one defines $E := (\epsilon, \bar{\epsilon})$, $S := (\sigma, \bar{\sigma})$, and $*|_{(\bigotimes^n \mathcal{A})_d} := *_n$, then it is straightforward to verify that $\{\mathcal{A}_d, \Phi, E, S\}$ is a Hopf algebra isomorphic to $\{\mathcal{A}, \phi, \epsilon, \sigma\}$ and, if equipped with $*$, satisfies the axioms of a $*$-Hopf algebra; we will call it a *nonstandard* $*$-realization of $\{\mathcal{A}, \phi, \epsilon, \sigma\}$.

Note that definition (3.20) is equivalent to

$$\begin{cases} \rho_n(*_n) := \gamma_n^{-1} \circ (\bigotimes^n *) \\ \bar{\rho}_n(*_n) := \gamma_n \circ (\bigotimes^n *) \end{cases} \tag{3.20}$$

and one can give to the above structure a more conventional form by noting that $\mathcal{A}$ endowed with either $\{\phi, \epsilon, \sigma, \rho(*)\}$ or $\{\bar{\phi}, \bar{\epsilon}, \bar{\sigma}, \bar{\rho}(*)\}$ forms a $*$ Hopf-algebra. Nevertheless, we prefer to present it as shown, since at the representation-theoretic level both the first and the second component of $(\bigotimes^n \mathcal{A})_d$ will be needed to act on two different vector-space realizations of the same Hilbert space (see section 4).

In section 4 we will clarify the relevance for representation theory of *nonstandard* realization of a $*$-Hopf algebra. Next, let us show that such a structure can be realized in a rather simple way in our case.

For the sake of brevity, let us set

$$P_2 := \Phi(p), \qquad \Lambda_2 := \left((\lambda p^{-1}\otimes\mathbf{1})\phi(p), (\lambda p\otimes\mathbf{1})\frac{1}{\bar\phi(p)}\right),$$

$$P_1 := (\mathbf{1}\otimes p, \mathbf{1}\otimes p) \qquad \Lambda_1 := \left(\frac{1}{\phi(p)}(p\otimes\lambda), \bar\phi(p)(p^{-1}\otimes\lambda)\right); \tag{3.21}$$

note that $\Phi(\Lambda) = \Lambda_1\Lambda_2$. Morally, P_2 would be the total momentum of two particles, P_1 the momentum of one particle (the one represented in the second tensor factor), Λ_i the corresponding step operators. It is easy to check that $P_1, \Lambda_1, P_2, \Lambda_2$ have well-defined commutation relations

$$[P_1, \Lambda_1]_q = 0 = [P_2, \Lambda_2]_q \qquad [\Lambda_1, \Lambda_2] = 0$$

$$[P_1, P_2] = 0 \qquad [P_1, \Lambda_2] = 0 \qquad [P_2, \Lambda_1] = 0, \tag{3.22}$$

i.e. that both the ρ_2 and $\bar\rho_2$ images of these (q)-commutators actually vanish as a consequence of relations (2.1). Moreover, $\rho_2(P_i^{\pm 1}), \rho_2(\Lambda_i^{\pm 1})$ $(i = 1, 2)$ form a set of generators of $\mathcal{A}\otimes\mathcal{A}$, and so do $\bar\rho_2(P_i^{\pm 1}), \bar\rho_2(\Lambda_i^{\pm 1})$. Using definition (3.20) we arrive at the complex conjugation $*_2$

$$P_2^{*_2} = P_2 \qquad P_1^{*_2} = P_1 \qquad \Lambda_1^{*_2} = \Lambda_1^{-1} \qquad \Lambda_2^{*_2} = \Lambda_2^{-1} \tag{3.23}$$

which is compatible with the algebra structure (3.23). In particular, we see that P_2, P_1 are real and make up a Cartan subalgebra of $(\mathcal{A}\otimes\mathcal{A})_d$.

One can now ask whether the above is the unique possible *nonstandard* $*$-realization of $\{\mathcal{A}, \phi, \epsilon, \sigma\}$. This amounts to investigating the uniqueness of γ. γ as an isomorphism with the abovementioned properties is not unique; but it is the only one which can be represented as an isomorphism of operators defined within the Hilbert space $\mathcal{H}_{\mu_a}\otimes\mathcal{H}_{\mu_b}$ (see the appendix), namely the only one which can be used for physical purposes. Therefore, the unique complete set of commuting observables (including P_2) for a 2-particle system is $\{P_2, P_1\}$, and there is no ambiguity in the physics which can be drawn from the above scheme.

4 Hilbert space representation of the $*$-algebra $\{(\mathcal{A}\otimes\mathcal{A})_d\}$

The rationale behind nostandard $*$-realizations of Hopf algebras (resp. of bialgebras) is that they can be used to find Hilbert space representations of them. The underlying idea is the following. Since an element $A \in (\mathcal{A}\otimes\mathcal{A})_d$ can be realized both by $\rho(A)$ and $\bar\rho(A)$, we look for an unbarred and a barred carrier space $\rho(\mathcal{U}), \bar\rho(\mathcal{U}) \subset \mathcal{H}_{\mu_a}\otimes\mathcal{H}_{\mu_b}$ for $\rho(A), \bar\rho(A)$ respectively to act upon. Of course, there should be a one-to-one correspondence between the unbarred and barred carrier spaces, in such a way that all physical quantities (eigenvalues of observables) are preserved under this mapping. The tricky point is that the barred and unbarred realization are used simultaneously in defining the " right " scalar product, namely the one for which hermitean conjugation is a realization of $*_2$.

Let $\mathcal{H}_{\mu_a}, \mathcal{H}_{\mu_b}$ be the carrier spaces of two Hilbert space representations of $\mathcal{A}$. We will consider the case when the subsystems consist of distinct particles and use a classical statistics; consequently the result will be different according to which particle we put in the first, second,... factor in the tensor product. We will deal with identical particles elsewhere. We look for $\rho(\mathcal{U}), \bar\rho(\mathcal{U}) \subset \mathcal{H}_{\mu_a}\otimes\mathcal{H}_{\mu_b}$ and a vector space isomorphism $\gamma : \rho(\mathcal{U}) \to \bar\rho(\mathcal{U})$ such that

$$\gamma\left[\rho(A)|\psi>\right] = \gamma[\rho(A)]\gamma(|\psi>) \equiv \bar\rho(A)\gamma(|\psi>), \qquad A \in (\mathcal{A}\otimes\mathcal{A})_d, \quad |\psi> \in \rho(\mathcal{U}). \tag{4.24}$$

Let $\mathcal{C}$ be a real Cartan subalgebra of $(\mathcal{A}\otimes\mathcal{A})_d$. Assume we have determined eigenvectors $|\varphi>, |\bar{\varphi}>\in \mathcal{H}_{\mu_a}\otimes\mathcal{H}_{\mu_b}$ of $\rho_2(\mathcal{C}), \bar{\rho}_2(\mathcal{C})$ respectively, with the *same weight*. Define the vector space

$$\mathcal{U} := \{(\rho_2(\alpha)|\varphi>, \bar{\rho}_2(\alpha)|\bar{\varphi}>) \ \mid \ \alpha\in(\mathcal{A}\otimes\mathcal{A})_d\} \subset (\mathcal{H}_{\mu_a}\otimes\mathcal{H}_{\mu_b}, \mathcal{H}_{\mu_a}\otimes\mathcal{H}_{\mu_b}). \tag{4.25}$$

Let $< \ \| \ >_{a\otimes b}$ be the ordinary scalar product in $\mathcal{H}_{\mu_a}\otimes\mathcal{H}_{\mu_b}$, i.e. if $\|f>=|f'>_a\otimes|f''>_b$, $\|g>=|g'>_a\otimes|g''>_b$ then $<f\|g>_{a\otimes b}:=<f'|g'>_a<f''|g''>_b$. It has the property that

$$<f\|u\; g>_{a\otimes b}=<u^{*\otimes *}f\|g>_{a\otimes b}, \qquad\qquad u\in\mathcal{A}\otimes\mathcal{A}. \tag{4.26}$$

Lemma 1 *The formula*

$$<U\|V>:=<\bar{u}\|v>_{a\otimes b}+<u\|\bar{v}>_{a\otimes b} \qquad\qquad \begin{cases}\|U>\equiv(\|u>,\|\bar{u}>)\in\mathcal{U}\\ \|V>\equiv(|v>,\|\bar{v}>)\in\mathcal{U}\end{cases} \tag{4.27}$$

defines a (formal) sesquilinear inner product in $\mathcal{U}$ such that

$$<U\|V>^*=<V\|U>, \tag{4.28}$$

and a $*_2$-representation of $(\mathcal{A}\otimes\mathcal{A})_d$ on $\mathcal{U}$.

Proof . Sesquilinearity $<a_iU_i, b_jV_j>=a_i^*b_j<U_i,V_j>$ $(a_i,b_j\in\mathbf{C})$ is trivial, and so is property (4.29). The easy but fundamental point is to show that hermitean conjugation w.r.t. the scalar product (4.28) is a realization of $*_2$ of $(\mathcal{A}\otimes\mathcal{A})_d$. To this end note that

$$\begin{aligned}<U\|AV>-<A^{*_2}U\|V> &\stackrel{(4.28),(3.19)}{=} <\bar{u}\|\rho(A)v>_{a\otimes b}-<[\rho(A)]^{*\otimes *}\bar{u}\|v>_{a\otimes b}\\ &+<u\|\bar{\rho}(A)\bar{v}>_{a\otimes b}-<[\bar{\rho}(A)]^{*\otimes *}u\|\bar{v}>_{a\otimes b}\stackrel{(4.27)}{=}0. \qquad\qquad \diamondsuit\end{aligned} \tag{4.29}$$

The above definition is formal, i.e. it makes sense only if the RHS of equation (4.28) (where an infinite sum can appear) is finite. This can be always achieved by suitably shrinking $\mathcal{U}$. However, we will impose on $\mathcal{U}$ a stronger requirement (which we have already stated in the inclusion relation in eq. (4.26)), namely that for each element $\|U>\equiv(\|u>,\|\bar{u}>)$ both $\|u>$ and $\|\bar{u}>$ have finite norm in $\mathcal{H}_{\mu_a}\otimes\mathcal{H}_{\mu_b}$ (it is immediate to verify that this implies that $\|U>$ has a finite norm w.r.t. the inner product (4.28): it follows from the positivity and finiteness of the norm of $\|u>-\|\bar{u}>$ in $\mathcal{H}_{\mu_a}\otimes\mathcal{H}_{\mu_b}$). This serves for $\gamma:\rho(\mathcal{U})\to\bar{\rho}(\mathcal{U})$ to map a subspace of $\mathcal{H}_{\mu_a}\otimes\mathcal{H}_{\mu_b}$ into a subspace of $\mathcal{H}_{\mu_a}\otimes\mathcal{H}_{\mu_b}$.

The last requirement that inner product (4.28) must fulfill, in order that it may be taken as a scalar product in $\mathcal{U}$, is its positivity. It will be proved below (Proposition 1).

Remark. Definition (4.28) has its source of inspiration in Ref. [2], see also section 6.

Now let us enforce this general idea to our toy model.

Remark. From now on we need to specify whether q is greater or smaller than 1. We will explicitly consider only the case $0<q<1$.

We choose $\mathcal{C}$ as the subalgebra generated by P_1, P_2; since both $\rho(P_1)=\mathbf{1}\otimes p$ and $\bar{\rho}(P_1)=\mathbf{1}\otimes p$ act only on the second tensor factor of $\mathcal{H}_{\mu_a}\otimes\mathcal{H}_{\mu_b}$, the eigenvectors $\|\varphi>,\|\bar{\varphi}>\in\mathcal{H}_{\mu_a}\otimes\mathcal{H}_{\mu_b}$ are to be searched in the form

$$\|\varphi_{n_1}>=|\varphi_{n_1}>_a\otimes|n_1>_b, \qquad\qquad \|\bar{\varphi}_{n_1}>=|\bar{\varphi}_{n_1}>_a\otimes|n_1>_b \tag{4.30}$$

with

$$\begin{cases} |\varphi_{n_1} >_a = \sum\limits_{n=-\infty}^{\infty} \varphi_{n,n_1} |n >_a \\ |\bar{\varphi}_{n_1} >_a = \sum\limits_{n=-\infty}^{\infty} \bar{\varphi}_{n,n_1} |n >_a \end{cases} \tag{4.31}$$

Consequently, $P_1 \| \varphi_{n_1} >= \mu_b q^{n_1} \| \varphi_{n_1} >$. We can assume without loss of generality that $q \leq |\frac{\mu_b}{\mu_a}| < 1$, because this can be always achieved after a shift of the integers l which label the states $|l >_i$ of either $\mathcal{H}_{\mu_a}$ or $\mathcal{H}_{\mu_b}$.

Let μ be the would-be eigenvalue in the equation $P_2 \| \varphi_{n_1} >= \mu \| \varphi_{n_1} >$; then in the unbarred representation the latter explicitly reads

$$\left[\sum_{n=-\infty}^{\infty} \varphi_{n,n_1} (\mu_a q^n - \mu) |n >_a + \varphi_{n,n_1} \mu_b q^m |n+1 >_a \right] \otimes |n_1 >_b = 0 \tag{4.32}$$

implying recursive relations

$$\varphi_{n,n_1} (\mu_a q^n - \mu) + \varphi_{n-1,n_1} \mu_b q^{n_1} = 0 \tag{4.33}$$

for the coefficients φ_{n,n_1}. Similarly for the coefficients $\bar{\varphi}_{n,n_1}$ we find the relations

$$\bar{\varphi}_{n,n_1} (\mu_a q^n - \mu) + \bar{\varphi}_{n+1,n_1} \mu_b q^{n_1} = 0. \tag{4.34}$$

Let us see investigate for which values of μ the vector $\| \varphi_{n_1} >$ has a positive and finite norm w.r.t. the product formally introduced in equation (4.28). In terms of the coefficients φ_{n,n_1}, $\bar{\varphi}_{n,n_1}$ this would-be norm reads

$$< \varphi_{n_1} \| \varphi_{n_1} >= \sum_{n=-\infty}^{\infty} \bar{\varphi}^*_{n,n_1} \varphi_{n,n_1} \quad + \quad c.\ c. \tag{4.35}$$

It is easy to realize that convergence of the above series for large n requires the existence of a $n_2 \in \mathbf{Z}$ such that $\mu = \mu_a q^{n_2}$, because otherwise

$$\begin{cases} \varphi_{n,n_1} = \varphi_{s,n_1} (\frac{\mu_b q^{n_1}}{\mu})^{n-s} \frac{1}{(\frac{\mu_a}{\mu} q^{s+1};q)_{n-s}} \\ \bar{\varphi}_{n,n_1} = \bar{\varphi}_{s,n_1} (\frac{\mu}{\mu_b q^{n_1}})^{n-s} (\frac{\mu_a}{\mu} q^{s+1};q)_{n-s} \end{cases} \qquad n \geq s, \tag{4.36}$$

for some $s \in \mathbf{Z}$, implying

$$\sum_{n=s}^{\infty} \bar{\varphi}^*_{n,n_1} \varphi_{n,n_1} = \bar{\varphi}^*_{s,n_1} \varphi_{s,n_1} \sum_{n=s}^{\infty} 1 = \infty. \tag{4.37}$$

We add the further label n_2 to label the corresponding solutions. Replacing this value in equations (4.34-35) we find

$$\varphi_{n,n_1,n_2} = \begin{cases} \varphi_{n_2,n_1,n_2} (\frac{\mu_b q^{n_1-n_2}}{\mu_a})^{n-n_2} \frac{1}{(q;q)_{n-n_2}} & if \quad n \geq n_2 \\ 0 & otherwise \end{cases}$$

$$\bar{\varphi}_{n,n_1,n_2} = \begin{cases} \bar{\varphi}_{n_2,n_1,n_2} (\frac{\mu_b q^{n_1-n_2}}{\mu_a})^{n_2-n} \frac{1}{(q^{-1};q^{-1})_{n_2-n}} & if \quad n \leq n_2 \\ 0 & otherwise. \end{cases} \tag{4.38}$$

We see that $|\bar{\varphi}_{n_1,n_2}>\in \mathcal{H}_{\mu_a}\ \forall n_1,n_2\in\mathbf{Z}$, whereas $|\varphi_{n_1,n_2}>\in\mathcal{H}_{\mu_a}$ only if $n_1\geq n_2$ (otherwise its norm is ∞; for $q>1$ the situation is exactly inverse).

Define

$$\|n_2,n_1>:=(|\varphi_{n_1,n_2}>_a\otimes|n_1>_b,|\bar{\varphi}_{n_1,n_2}>_a\otimes|n_1>_b) \tag{4.39}$$

and

$$\mathcal{U}^{phys}:=Span_{\mathbf{C}}\{\|n_2,n_1>\quad,\quad n_1-n_2\geq 0\}. \tag{4.40}$$

According to definition (4.28), the square norm of $\|n_2,n_1>$ is equal to $\bar{\varphi}^*_{n_2,n_1,n_2}\varphi_{n_2,n_1,n_2}$ In order to get a positive norm it is sufficient to take the latter quantity positive. We will choose $\bar{\varphi}_{n_2,n_1,n_2}=1=\varphi_{n_2,n_1,n_2}$. Consequently,

Proposition 1 *The space of " physical " two particle states $\mathcal{U}^{phys}$ is a pre-Hilbert space with scalar product (4.28), and $\mathcal{B}_2:=\{\|n_2,n_1>\quad,\quad n_1-n_2\geq 0\}$ is an orthonormal basis, such that*

$$\begin{array}{ll} P_1\|n_2,n_1>=\mu_b q^{n_1}\|n_2,n_1> & P_2\|n_2,n_1>=\mu_a q^{n_2}\|n_2,n_1> \\ \Lambda_1^{\pm 1}\|n_2,n_1>=\|n_2,n_1\pm 1> & \Lambda_2^{\pm 1}\|n_2,n_1>=\|n_2\pm 1,n_1>. \end{array} \tag{4.41}$$

Moreover

$$\begin{cases} |\varphi_{n_1,n_2}>=\sum\limits_{n=0}^{\infty}(\frac{\mu_b q^{n_1-n_2}}{\mu_a})^n\frac{1}{(q,q)_n}|n_2+n> \\ |\bar{\varphi}_{n_1,n_2}>=\sum\limits_{n=0}^{\infty}(\frac{\mu_b q^{n_1-n_2}}{\mu_a})^n\frac{1}{(q^{-1},q^{-1})_n}|n_2-n>. \end{cases} \tag{4.42}$$

Remark. The spectrum of the total momentum P_2 is $\{\mu_a q^n\}_{n\in\mathbf{Z}}$, the same as that of $p\otimes\mathbf{1}$; the momentum scale μ_a is the same as that of the first particle. The eigenvectors of $\rho(P_2),\bar{\rho}(P_2)$ are *superpositions* of eigenvectors of $p\otimes\mathbf{1}$, i.e. a sort of interaction between the first and second particle arises.

The name " physical " is due to the fact that, as a consequence of its definition, $\mathcal{U}_{phys}$ is characterized by the physical condition that on each eigenvector of P_1,P_2 (recall that P_2 represents the total momentum of the two-particle system, whereas P_1 the momentum of the second particle)

$$|<P_2>|\ >\ |<P_1>|\qquad if\ \ \frac{\mu_a}{\mu_b}>0; \tag{4.43}$$

compare with eq.'s (2.8).

Finally, note that

$$\Lambda_1^{-1}\|n,n>\notin\mathcal{U}^{phys}\qquad\qquad \Lambda_2\|n,n>\notin\mathcal{U}^{phys}, \tag{4.44}$$

since the corresponding unbarred series have infinite norm in $\mathcal{H}_{\mu_a}\otimes\mathcal{H}_{\mu_b}$.

5 Generalization to n particles

Let us define for the sake of brevity

$$P_i:=(\rho_n(P_i),\bar{\rho}_n(P_i))\qquad\qquad \Lambda_i:=(\rho_n(\Lambda_i),\bar{\rho}_n(\Lambda_i))$$

$$\begin{cases} \rho_n(P_i):=\underbrace{\mathbf{1}\otimes...\otimes\mathbf{1}}_{(n-i)\quad times}\otimes\phi_i(p) \\ \bar{\rho}_n(P_i):=\underbrace{\mathbf{1}\otimes...\otimes\mathbf{1}}_{(n-i)\quad times}\otimes\bar{\phi}_i(p) \end{cases} \tag{5.1}$$

$$\begin{cases} \rho_n(\Lambda_i) := \underbrace{\mathbf{1}\otimes ... \otimes \mathbf{1}}_{(n-i-1)\quad times} \otimes \frac{\mathbf{1}\otimes\phi_i(p)}{\phi_{i+1}(p)}(p\otimes p^{-1}\lambda \otimes \underbrace{\mathbf{1}\otimes ... \otimes \mathbf{1}}_{(i-1)\quad times}) \\ \bar\rho_n(\Lambda_i) := \underbrace{\mathbf{1}\otimes ... \otimes \mathbf{1}}_{(n-i-1)\quad times} \otimes \frac{\bar\phi_{i+1}(p)}{\mathbf{1}\otimes\phi_i(p)}(p^{-1}\otimes p\lambda \otimes \underbrace{\mathbf{1}\otimes ... \otimes \mathbf{1}}_{(i-1)\quad times}) \end{cases} \qquad if \quad n > i.$$

$$\begin{cases} \rho_n(\Lambda_n) := \phi_n(p)(\lambda p^{-1} \otimes \underbrace{\mathbf{1}\otimes ... \otimes \mathbf{1}}_{(n-1)\quad times}) \\ \bar\rho_n(\Lambda_n) := \frac{1}{\bar\phi_n(p)}(\lambda p \otimes \underbrace{\mathbf{1}\otimes ... \otimes \mathbf{1}}_{(n-1)\quad times}) \end{cases} \tag{5.2}$$

$((\phi_1(p) \equiv p)$. We denote by P_i^{-1}, Λ_i^{-1} the inverses of P_i, Λ_i. Then

Proposition 2 $P_i, \Lambda_i, \;\; i,j = 1,2,...,n,$ *have well-defined commutation relations*

$$[P_i, P_j] = 0 \qquad [\Lambda_i, \Lambda_j] = 0 \qquad P_i\Lambda_j = q^{\delta_i^j}\Lambda_j P_i \tag{5.3}$$

and therefore form together with P_i^{-1}, Λ_i^{-1} *a set of generators of* $(\bigotimes^n \mathcal{A})_d := \{(\alpha, \gamma_n(\alpha)) \;\; | \;\; \alpha \in \bigotimes^n \mathcal{A}\}$, $\gamma_n := \bar\rho_n \circ \rho_n^{-1}$. *Moreover definition (3.20) introduces a complex conjugation* $*_n$ *which is compatible with the algebra relations (5.3):*

$$P_i^{*_n} = P_i \qquad\qquad \Lambda_i^{*_n} = \Lambda_i^{-1} \tag{5.4}$$

Proof The proof of the commutation relations (5.3) is recursive. Assume that they hold when $n = m$. The corresponding ρ images read:

$$[\rho_m(P_i), \rho_m(P_j)] = 0 \qquad i,j = 1,...,m \tag{5.5}$$

Let

$$\phi_{m,l} := (\underbrace{id\otimes ... \otimes id}_{(m-l)\quad times} \otimes \phi \otimes \underbrace{id\otimes ... \otimes id}_{(l-1)\quad times}) \qquad 1 \le l \le m; \tag{5.6}$$

now note that from definition (5.1) and equation (1.3) it follows

$$\phi_{m,l}(\rho_m(P_i)) = \begin{cases} \rho_{m+1}(P_i) & if \quad l > i \\ \rho_{m+1}(P_{i+1}) & if \quad l \le i \end{cases} \tag{5.7}$$

Therefore applying the operators $\phi_{m,l}$ to both sides of equations $(5.3)_1$ in the case $n = m$ we get the ρ images of equations $(5.3)_1$ in the case $n = m+1$. The remaining relations $(5.3)_2, (5.3)_3$ are most simply proved by direct calculation, after noting that

$$\left[\rho_n(P_i), \; \underbrace{\mathbf{1}\otimes ... \otimes \mathbf{1}}_{(n-j-1)\quad times} \otimes p \otimes p^{-1}\lambda \otimes \underbrace{\mathbf{1}\otimes ... \otimes \mathbf{1}}_{(j-1)\quad times}\right]_{q^{\delta_i^j}} = 0. \tag{5.8}$$

The proof is similar for the $\bar\rho$-images. $\diamondsuit$

The procedure described in section 4 can be iterated in a straightforward way to yield a $*_n$-representation of $(\bigotimes^n \mathcal{A})_d$ on a Hilbert space $\mathcal{U}^{phys}_{\mu_n,...,\mu_1} \subset (\bigotimes_{i=1}^n \mathcal{H}_{\mu_{n-i+1}}, \bigotimes_{i=1}^n \mathcal{H}_{\mu_{n-i+1}})$ which is the linear span of a set of vectors $\mathcal{B}_n := \{\|n_n,...,n_1>, \quad n_{i+1} \le n_i\}$ where, for $i = 1,2,...,n$,

$$P_i\|n_n,...,n_1>= \mu_i q^{n_i}\|n_n,...,n_1> \qquad\qquad \Lambda_i^{\pm 1}\|n_n,...,n_1>=\|n_n,...,n_i\pm 1,...,n_1>; \tag{5.9}$$

moreover $\mathcal{B}_n$ is an orthonormal basis of $\mathcal{U}^{phys}_{\mu_n,...,\mu_1}$ w.r.t. the scalar product (4.28).

It goes as follows: assume that one has been found for $n = m$, and that the mass scale of P_m (the total momentum of m particles) is μ_m (i.e. the mass scale of the Hilbert space which is the first tensor factor in $\bigotimes_{i=1}^{m} \mathcal{H}_{\mu_{m-i+1}}$). Then we introduce vectors $\|n_{m+1}, n_m, ..., n_1 >$ by replacing in formula (4.31) to μ_a, n_2 respectively μ_{m+1}, n_{m+1}, and to the vector $|n_1 >_b$ the vector $\rho(\|n_m, n_{m-1}, ..., n_1 >)$, and similarly for $\bar{\rho}_{m+1}$, in other words we set

$$\begin{cases} \rho_{m+1}(\|n_{m+1}, n_m, ..., n_1 >) := |\varphi_{n_m,n_{m+1}} >_{m+1} \otimes \rho(\|n_m, ..., n_1 >) \\ \bar{\rho}_{m+1}(\|n_{m+1}, n_m, ..., n_1 >) := |\bar{\varphi}_{n_m,n_{m+1}} >_{m+1} \otimes \bar{\rho}(\|n_m, ..., n_1 >) \end{cases} \qquad \|n_m, ..., n_1 > \in \mathcal{B}_m \tag{5.10}$$

and

$$\|n_{m+1}, n_m, ..., n_1 >= (\rho_{m+1}(\|n_{m+1}, n_m, ..., n_1 >), \bar{\rho}_{m+1}(\|n_{m+1}, n_m, ..., n_1 >)) \tag{5.11}$$

The set $\mathcal{B}_{m+1} := \{\|n_{m+1}, n_m, ..., n_1 >, \quad n_{i+1} \leq n_i\}$ is a set of eigenvectors of P_i, $i = 1, 2, ..., m+1$ with eigenvalues $\mu_i q^{n_i}$, orthonormal w.r.t. the product (4.28) where in the RHS $\mathcal{H}_a$ stands for the space $\mathcal{H}_{\mu_{m+1}}$ and $\mathcal{H}_b$ for the space $\mathcal{U}_{\mu_m,...,\mu_1}$. Moreover, a glance at definition (3.20) (of $*_{m+1}$) will convince the reader that $*_{m+1}$ on P_i, Λ_i $i = 1, 2, ..., m+1$ acts as hermitean conjugation w.r.t. the scalar product (4.28). Finally

$$P_i\|n_{m+1}, ..., n_1 >= \mu_i q^{n_i}\|n_{m+1}, ..., n_1 > \qquad \Lambda_i^{\pm 1}\|n_{m+1}, ..., n_1 >= \|n_{m+1}, ..., n_i \pm 1, ..., n_1 >, \tag{5.12}$$

as claimed.

6 Configuration space realization

It is possible to realize the Hilbert space $\mathcal{H}_\mu$ as a pair of subspaces of $Fun(\mathbf{R}_q)$ and the algebra $\mathcal{A}$ as a subalgebra of the differential algebra $Diff(\mathbf{R}_q)$ on $\mathbf{R}_q$, in other words in terms of q-differential operators acting on $Fun(\mathbf{R}_q)$. The $*$ is realized as the one of $Diff(\mathbf{R}_q)$, and the scalar product of $\mathcal{H}_\mu$ through a q-integral [7] . The procedure is a word by word translation of the configuration space realization of the harmonic oscillator [2] and of the free particle [3] on $\mathbf{R}_q^N$; this is the source of inspiration for the main idea of the present work. Moreover, the construction of multi-particle systems can be done as above and is best understood in terms of braided configuration spaces.

Let $Fun(\mathbf{R}_q)$ be the unital algebra generated by x, $Diff(\mathbf{R}_q)$ the unital algebra generated by ∂, x with derivation relations

$$\partial x = 1 + qx\partial. \tag{6.1}$$

The dilaton $\tilde{\lambda}$ is defined by

$$\tilde{\lambda} := [\partial, x] = 1 + (q-1)x\partial \qquad \Rightarrow \qquad \tilde{\lambda}x = qx\tilde{\lambda}, \qquad \tilde{\lambda}\partial = q^{-1}\partial\tilde{\lambda}. \tag{6.2}$$

If we introduce a complex conjugation $\star$ by requiring that x is real then,

$$x^\star = x, \qquad \partial^\star = -q^{-1}\bar{\partial} \qquad \Rightarrow \tilde{\lambda}^\star = q^{-1}\tilde{\lambda}^{-1}, \tag{6.3}$$

where $\bar{\partial}$ is the barred derivative

$$\bar{\partial} := \tilde{\lambda}^{-1}\partial, \qquad \bar{\partial}x = 1 + q^{-1}x\bar{\partial}. \tag{6.4}$$

Contrary to the abstract treatment of section 4, here it is convenient also in the case of a one-particle system to give a pair of realizations both of $\mathcal{A}$ and of $\mathcal{H}_\mu$, we will call them $\tilde{\rho_1}(\mathcal{A}), \tilde{\rho_1}(\mathcal{H}_\mu)$ and $\tilde{\bar{\rho}}_1(\mathcal{A}), \tilde{\bar{\rho}}_1(\mathcal{H}_\mu)$ respectively. We start from

$$\begin{cases} \tilde{\rho_1}(p) = i\partial \\ \tilde{\bar{\rho}}_1(p) = i\bar{\partial} \end{cases} \qquad \begin{cases} \tilde{\rho_1}(\lambda^{\pm 1}) = \tilde{\lambda} \\ \tilde{\bar{\rho}}_1(\lambda^{\pm 1}) = q^{\pm 1}\tilde{\lambda}^{\pm 1} \end{cases}. \tag{6.5}$$

Setting $\mathcal{A}'_d := \{(\tilde{\rho_1}(a), \tilde{\bar{\rho}}_1(a)) \;\mid\; a \in \mathcal{A}\} \approx \mathcal{A}$, one can realize the $*$-involution of $\mathcal{A}$ on $\mathcal{A}'_d$ by setting

$$(\tilde{\rho_1}(a^*), \tilde{\bar{\rho}}_1(a^*)) = \tau \circ (\star, \star)[(\tilde{\rho_1}(a), \tilde{\bar{\rho}}_1(a))]. \tag{6.6}$$

The $\tilde{\rho_1}, \tilde{\bar{\rho}}_1$ images of the vectors $|n>$ are determined as functions of x which satisfy the $\tilde{\rho_1}, \tilde{\bar{\rho}}_1$ image of the eigenvalue equations $p|n>= \mu q^n|n>$ and of the " step " equation $\Lambda^{\pm 1}|n>=|n \pm 1>$:

$$\begin{cases} \tilde{\rho_1}(|n>) := e_q[-i\mu q^n x] \\ \tilde{\bar{\rho}}_1(|n>) := q^n e_{q^{-1}}[-i\mu q^n x] \end{cases} \tag{6.7}$$

where

$$e_q[Z] := \sum_{l=0}^{\infty} \frac{Z^l}{l_q!} \qquad l_q := \frac{1-q^l}{1-q}; \tag{6.8}$$

$\tilde{\rho_1}, \tilde{\bar{\rho}}_1$ are extended on $\mathcal{A}$ as algebra isomorphisms, on $\mathcal{H}_\mu$ as vector space isomorphisms. Therefore to each $|\psi>\in \mathcal{H}_\mu$ there will correspond two functions $\psi(x) := \tilde{\rho_1}(|\psi>)$ and $\bar{\psi}(x) := \tilde{\bar{\rho}}_1(|\psi>)$. The scalar product in $\mathcal{H}_\mu$ will be realized through algebraic q-integration [7, 6]

$$<\psi'|\psi>= \int d_q x[\bar{\psi}'(x)^\star \psi(x) + (\psi')^\star(x)\bar{\psi}(x)] \tag{6.9}$$

after fixing its normalization properly. Due to the properties of q-Stokes theorem of q-integration

$$\int d_q x \partial f(x) = 0 = \int d_q x \bar{\partial} f(x), \tag{6.10}$$

definition (6.9) is compatible with hermitean conjugation of the $*$-representation of $\mathcal{A}$ on $\mathcal{H}_\mu$. All the treatment done in the abstract case applies here too, provided we modify the definitions of $(\bigotimes^n \mathcal{A})_d$ into $(\bigotimes^n \mathcal{A})'_d := (\bigotimes^n \tilde{\rho_1}, \bigotimes^n \tilde{\bar{\rho}}_1)[(\bigotimes^n \mathcal{A})_d]$. One can easily verify that

$$\begin{cases} \tilde{\rho_1}(|\varphi_{n_1,n_2}>)(x) =: \varphi_{n_1,n_2}(x) = \sum_{l=0}^{\infty} [i\mu_a q^{n_2}(q-1)x]^l \frac{(\frac{q^{n_1-n_2}\mu_b}{\mu_a};q)_l}{(q;q)_l} \\ \tilde{\rho_1}(|\bar{\varphi}_{n_1,n_2}>)(x) =: \bar{\varphi}_{n_1,n_2} = \sum_{l=0}^{\infty} [i\mu_a q^{n_2}(q^{-1}-1)x]^l \frac{(\frac{q^{n_1-n_2-1}\mu_b}{\mu_a};q^{-1})_l}{(q^{-1};q^{-1})_l}, \end{cases} \tag{6.11}$$

by solving the corresponding eigenvalue equations in $Fun(\mathbf{R}_q)$.

7 Classical limit

In order that the toy-model considered in this work can be considered as physically realistic, it should describe a system of one or more free particles on a continuous line in the limit (understood in some reasonable sense) $q \to 1^-$.

Let us denote by $|n>_q$ the orthonormal basis vectors of $\mathcal{H}_\mu$ for a given $q<1$, $p|n>_q= \mu q^n|n>_q$. For any fixed n $\lim_{q\to 1^-} \mu q^n = \mu$, i.e. all eigenvalues collapse (but *not* uniformly)

to μ, therefore the set $\{\lim_{q\to 1^-}|n>_q\}_{n\in\mathbf{Z}}$ is no generalized basis of the representation space of the undeformed 1-dimensional translation+dilatation group, and, as already noticed in ref. [3], the limit $q\to 1^-$ cannot be understood a naive sense.

A more adequate notion of such a limit seems the following (see also Ref. [3]). For each nonnormalizable eigenvector (i.e. distribution) $|\pi>_c$ of the classical momentum p, $p|\pi>_c=\pi|\pi>_c$, there exists a function $\tilde{n}(\pi,q)$ such that $|\pi>_c=\lim_{q\to 1^-}|n(\pi,q)>_q$, in the sense of convergence of the eigenvalues. Such a problem admits the following solution

$$\tilde{n}(\pi,q)=[log_q(\frac{\pi}{\mu})], \tag{7.1}$$

where $[a]$ denotes the integral part of $a\in\mathbf{R}$. In fact the eigenvalue of p on $|\tilde{n}(\pi,q)>_q$ goes to π in the limit $q\to 1^-$, and the expression $\lim_{q\to 1^-}|n(\pi,q)>_q$ could be given a sense as a distribution on $\mathcal{S}(\mathbf{R})$. Therefore we can say that the toy-model is appropriate for an approximate description of *one* ordinary free particle, provided we choose q sufficiently close to 1.

When considering *many*-particle systems, we have seen that an unavoidable sort of interaction arises as a consequence of the inthrinsic braided character of the total momentum. Let us stick for simplicity to the case of two particles. In section 4 we have seen that the eigenvectors $\|n_1,n_2>_q$ of P_1,P_2 is the tensor product of two one-particle states, the second is a free one-particle state $|n_1>_q$, whereas the first is a superposition of different free one-particle states; in the ρ realization it is denoted by $|\varphi_{n_1,n_2}>_q$. In order that the two-particle states $\|n_1,n_2>_q$ are realistic descriptions of the classical free two-particle states $\|\pi_1,\pi_2>_c$ (defined by $P_i\|\pi_1,\pi_2>_c=\pi_i\|\pi_1,\pi_2>_c$, $\pi_2>\pi_1$), we should check that in the unbarred representation in the limit $q\to 1^-$ the superposition $|\varphi_{\tilde{n}_1,\tilde{n}_2}>_q$ ($\tilde{n}_i(q,\pi_i):=[log_q(\frac{\pi_i}{\mu_i})]$), becomes a distribution with well-defined momentum $\pi_2-\pi_1$, i.e. that the interaction effect disappears. Let us check that this is the case. Let $\pi:=\frac{\pi_1}{\pi_2}$ ($\pi<1$). Then

$$|\varphi_{\tilde{n}_1,\tilde{n}_2}>_q=\sum_{n=0}^{\infty}c_n|n+n_2(q,\pi_2)> \qquad\qquad c_n:=\frac{(\pi r(q,\pi))^n}{(q;q)_n} \tag{7.2}$$

where the rest $r(q,\pi):=(\frac{\mu_b q^{\tilde{n}_1-\tilde{n}_2}}{\pi\mu_a})$ satisfies the relation $q^2\le r(q,\pi)<1$ ($\lim_{q\to 1^-}r=1$). The function $(\pi r(q,\pi))^n$ decreases with n, the function $\frac{1}{(q;q)_n}$ increases with n, therefore there is a value $n=\bar{n}(q,\pi)$ for which c_n is maximum. To determine $\bar{n}$ (up to an error of 1) we consider n as a continuous variable and look for the solution of the equation $\frac{\partial}{\partial n}c_n=0$. We are actually interested in the asymptotic solution in the limit $q\to 1^-$. In order to properly differentiate the function in the denominator we note that $\frac{1}{(q,q)_n}:=\frac{(q^{n+1};q)_\infty}{(q;q)_\infty}$. We find

$$0\equiv\frac{\partial}{\partial n}c_n=c_n[ln(\pi)-ln(q)g(n,q)], \tag{7.3}$$

where

$$g(n,q):=\sum_{k=0}^{\infty}\frac{q^{n+k+1}}{1-q^{n+k+1}}=\sum_{k=0}^{\infty}q^{n+k+1}\sum_{l=0}^{\infty}q^{(n+k+1)l}=\sum_{l=0}^{\infty}q^{(n+1)(l+1)}\sum_{k=0}^{\infty}q^{k(l+1)}=\sum_{l=0}^{\infty}\frac{q^{(n+1)(l+1)}}{1-q^{l+1}}. \tag{7.4}$$

Hence

$$ln(q)g(n,q)\overset{q\to 1^-}{\approx}=-\sum_{l=0}^{\infty}\frac{q^{(n+1)(l+1)}}{(l+1)_q}\overset{q\to 1^-}{\approx}ln(1-q^{n+1}) \tag{7.5}$$

since the q-deformed integer $l_q := \frac{q^l-1}{q-1}$ goes to $l \in \mathbf{Z}$ in the limit $q \to 1^-$; therefore the searched solution $\bar{n}(q,\pi)$ is such that $q^{\bar{n}} \stackrel{q\to 1^-}{\approx} 1-\pi$. This implies that in the superposition $|\varphi_{\tilde{n}_1,\tilde{n}_2}>_q$ the eigenvector with the eigenvalue $p = \pi_2(1-\pi) = \pi_2 - \pi_1$ is enhanced. Furthermore, by setting $f(u) := c_n|_{n=[log_q(1-u)]}$ one could show that $f(u)$ is more and more peaked around $u = \pi$ in the limit $q \to 1^-$. This means that in that limit $|\varphi_{\tilde{n}_1,\tilde{n}_2}>_q$ goes to the distribution describing a free particle with momentum $\pi_2 - \pi_1$ on $\mathbf{R}$, as claimed.

8 Appendix

In this appendix we clarify in which sense for a n-particle system the set $\{P_1, P_2, ..., P_n\}$ considered in section 5 is the only admissible complete set of commuting observables including the total momentum $\Phi_n(p)$. For the sake of brevity we stick to the case $n = 2$.

Let

$$a := \phi(p), \qquad b := \mathbf{1} \otimes p, \qquad c := p \otimes \lambda, \qquad d := \lambda \otimes \lambda. \tag{8.6}$$

It is immediate to realize that a, b, c, d and their inverses are a complete set of independent generators of $\mathcal{A} \otimes \mathcal{A}$, and $[a, b] = 0 = [a, c]$. The unbarred image of any observable V independent of $\Phi(p)$ and commuting with it is therefore a function of b, c only. The subalgebra $\mathcal{V}$ generated by b, c can be decomposed as $\mathcal{V} = \bigoplus_{n\in\mathbf{Z}} (\mathcal{V}_1)^n$, where $\mathcal{V}_1$ is its component satisfying the relation $[\mathcal{V}_1, d]_q = 0$, i.e. with natural dimension 1. Thus we can search $v = \rho(V)$ within $\mathcal{V}_1$ without loss of generality. $\mathcal{V}_1$ is a subspace spanned by $c^{-j}b^{j+1}$; therefore v has to be a combination (which we will assume to be finite)

$$v = \sum_{j=J_0}^{J_1} v_j c^{-j} b^{j+1} \in \mathcal{V}_1. \tag{8.7}$$

The pair $V := (v, v^{*\otimes *})$ is a real element of $(\mathcal{A} \otimes \mathcal{A})_d$; we will say that V is admissible (and consequently can be chosen as an observable, beside $\Phi(p)$) if the eigenvectors $|\psi>$ of $\phi(p), v$ are elements of (i.e. normalizable in) $\mathcal{H}_{\mu_a} \otimes \mathcal{H}_{\mu_b}$. We ask now for which choice of the coefficients v_i V is admissible.

Proposition 3 *$V := (v, v^*)$ is admissible only if $v \propto b$; in other words P_1 is the only commuting observable which we can add to $P_2 = \Phi(p)$.*

Proof. Reasoning as in section 4, it is easy to check that, independently of the choice of v, the spectrum of a is $\{\mu_a q^n\}_{n\in\mathbf{Z}}$, and

$$a|\psi_{n_2}>= \mu_a q^{n_2}|\psi_{n_2}> \qquad \Rightarrow \qquad |\psi_{n_2}>:= \sum_{m=0}^{\infty} c_{n_2,m} \left[\sum_{n=0}^{\infty} \frac{(\frac{\mu_b}{\mu_a}q^m)^n}{(q;q)_n}|n+n_2>\right] \otimes |m+n_2> . \tag{8.8}$$

The requirement of normalizability in $\mathcal{H}_{\mu_a} \otimes \mathcal{H}_{\mu_b}$ of $|\psi_{n_2}>$ implies

$$\sum_{m=0}^{\infty} |c_{n_2,m}|^2 < \sum_{m=0}^{\infty} |c_{n_2,m}|^2 d_m =<\psi_{n_2}||\psi_{n_2}>_{a\otimes b}< \infty, \qquad (here\ \ d_m := \sum_{n=0}^{\infty}\left[\frac{(\frac{\mu_b}{\mu_a}q^m)^n}{(q;q)_n}\right]^2 > 1); \tag{8.9}$$

consequently there exists a constant $C > 0$ such that $|\frac{c_{n_2,m+h}}{c_{n_2,m}}| \leq C$ if $h > 0$, m is large enough and $c_{n_2,m+h} \neq 0 \neq c_{n_2,m}$. Now assume that $|\psi_{n_2}>$ is also an eigenvector of v with eigenvalue

μ. By easy computations one shows that the eigenvalue equation $(v-\mu)|\psi_{n_2}>=0$ amounts to

$$\theta(k-J_1)\sum_{j=\max(J_0,-k)}^{J_1}\hat{v}_j c_{n_2,j+k}q^{k(j+1)+n_2}=\theta(k)\frac{\mu}{\mu_b}c_{n_2,k}, \tag{8.10}$$

where $\hat{v}_j := v_j(\frac{\mu_b}{\mu_a})^j q^{j(j+1)}$ and $\theta(s)=\begin{cases}0 & if \;\; s<0\\ 1 & if s\geq 1\end{cases}$.

It is easy to show that if $J_1\neq 0$ then $c_{n_2,k}=0\ \forall k\geq 0$. In fact, if $J_1<0$, then choosing first $k=0$, then $k=1$, $k=2$ etc, we iteratively show that the LHS in equation (8.10) vanishes, and so does $c_{n_2,k}$. Similarly, if $J_1>0$, then choosing first $k=-J_1$, then $k=-J_1+1$, $k=-J_1+2$ etc, we iteratively show that the RHS in equation (8.10) vanishes, and so does $c_{n_2,k}$. Therefore a nontrivial solution is possible only if $J_0\leq J_1=0$. Under these circumstances one can rewrite equation (8.10) as follows:

$$c_{n_2,k}\left(\frac{\mu}{\mu_b}-q^{k+n_2}\hat{v}_0\right)=\sum_{j=J_0}^{-1}\hat{v}_j c_{n_2,j+k}q^{k(j+1)+n_2} \tag{8.11}$$

Unless $J_0=0$, the RHS is different from zero and in the limit $k\to\infty$ its dominant contribution comes from $j=J_0$, because of the diverging factor $q^{k(j+1)}$ and the bound $|\frac{c_{n_2,k+j+h}}{c_{n_2,k+j}}|\leq C$:

$$c_{n_2,k}\left(\frac{\mu}{\mu_b}-q^{k+n_2}\hat{v}_0\right)\stackrel{q\to 1^-}{\approx}\hat{v}_{J_0}c_{n_2,J_0+k}q^{k(J_0+1)+n_2}; \tag{8.12}$$

consequently, $|c_{n_2,k}|$ does not converge to zero because of the divergent factor $q^{k(J_0+1)}$, against the hypothesis. Therefore, it must be $J_0=J_1=0$, namely, we have proved the claim . $\diamondsuit$

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